Classical Systems and Representations of (2+1) Newton–Hooke Symmetries

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Abstract

A study of the Newton–Hooke kinematical groups in (2+1) dimensions is presented revealing the physical interest of these symmetries. A complete classification of both classical and quantum elementary systems is achieved by explicit computation of coadjoint orbits and unitary irreducible representations of extended groups. In addition, we present an application example of quantization *la Moyal* of a classical system using the Stratonovich–Weyl correspondence and also give some ideas about a second central extension, which did not appear in the (3+1)–dimensional case.

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1 Introduction

The Newton–Hooke groups, $NH_{\pm}$, were introduced some years ago by Bacry and Lvy–Leblond [1] as Newton groups. Later, Derome and Dubois [2] carried out a complete study of these groups in the (3+1)–dimensional (D) case

giving them the name of Hooke groups. These groups appeared while looking for all the possible realizations of the relativity principle. So, they are kinematical groups, such as Galilei and Poincar ones. They can be considered non-relativistic cosmological groups characterized by a temporal constant τ which determines the characteristic time of the universe stable under the group action.

It is well–known in Physics the fact that restriction to one spatial dimension implies a great simplification in mathematical models to be used which entails a substantial decrease in calculations. However, this is not the case when one consider 2D models, in which case the symmetry of the systems results in a wide variety of concepts.

Classical and quantum systems associated with (1+1)D kinematical groups have been studied in relation with quantization problem [3]–[7]. Concerning (2+1)D kinematical groups, Ref. [8] follows the same patterns for the Galilei group and Refs. [9]–[12] are only focused on representation theory for Galilei or Poincar groups.

In this context, the intrinsic interest of Newton–Hooke $(2+1)$ groups (related with the harmonic oscillator and systems in expansion as we will see later on) is enlarged with a rich structure of central extensions and a wide set of different classical (coadjoint orbits) and quantum (irreducible representations) elementary systems.

In the framework of Inn–Wigner contractions [13], the groups we deal with here correspond to space–velocity kind contractions of the de Sitter or Anti–de Sitter groups, $dS_{\pm}$, and leads to Galilei group through a space–time contraction. This property is important for the sake of interpretation as we will see in section 6. In these sense, $NH_{\pm}$ groups can be considered as non-relativistic cosmological kinematical groups. On the other hand, the study of Hooke cosmology reveals that the main difference between Newton–Hooke and Galilei groups is the existence of a harmonic oscillator (NH_{-}) or expansion (NH_{+}) potential which make these groups can be considered as dynamical groups of galilean systems. Characteristic time τ can then be interpreted in terms of the inverse of Hubble's constant [2] for the expanding universe or associated to the "period" for the oscillating case.

The paper is organized as follows. In section 2 we summarize the basic definitions of the $NH(2+1)$ groups as well as their corresponding Lie algebras and central extensions. Section 3 is devoted to the construction and classification of classical elementary systems associated with these groups via the coadjoint action and it presents a simple example of physical interpretation. In section 4 we develop the theory of representations of these Lie groups

in order to get the quantum elementary systems and their classification. In section 5 we give to the reader the basic references concerning Moyal's quantization by means of the Stratonovich–Weyl correspondence and present an example of application for elementary systems obtained in previous sections. Finally, section 6 is intended to discuss some results about extensions and expose our conclusions.

2 The (2+1) Newton–Hooke Group

We will focus our attention on $NH_-(2+1)$, the oscillating group denoted NH henceforth, the study for NH_+ is similar and can be achieved by replacing trigonometric functions by hyperbolic ones in group law. Thus, NH can be considered the group of transformations of space–time acting on the point of coordinates $(t, \mathbf{x})$ by

$$(t', \mathbf{x}') = g(t, \mathbf{x}) = \big(t + b,\ \mathbf{x}^\phi + \mathbf{v}\tau \sin\frac{t}{\tau} + \mathbf{a}\cos\frac{t}{\tau}\big), \tag{2.1}$$

where $\mathbf{x}^\phi = R(\phi)\mathbf{x}$ stands for the two–dimensional vector $\mathbf{x}$ rotated an angle ϕ $\big(R(\phi) \in SO(2)\big)$. Explicitly, $\mathbf{x}^\phi = \left(\begin{smallmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{smallmatrix}\right)\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$.

The group elements are parametrized by

$$g = (b, \mathbf{a}, \mathbf{v}, \phi), \qquad b, \phi \in \mathbb{R};\ \mathbf{a}, \mathbf{v} \in \mathbb{R}^2, \tag{2.2}$$

$\mathbf{a}$, b being the parameters of space–time translations, $\mathbf{v}$ that of pure NH inertial transformations (boosts) and ϕ that of rotations.

The constant τ symbolizes the proper or characteristic time we mentioned before [2] and is not reabsorbed in time units for the sake of physical interpretation.

The corresponding Lie algebra, $\mathcal{NH}$, has dimension six and is spanned by the infinitesimal generators of time translations, H, space translations, $\mathbf{P} \equiv (P_1, P_2)$, boosts, $\mathbf{K} \equiv (K_1, K_2)$, and rotations around an axis perpendicular to the plane, J. Their non–vanishing commutation relations are

$$\begin{aligned} &[J, P_i] = \epsilon_{ij} P_j, \qquad [P_i, H] = \tfrac{-1}{\tau^2} K_i, \\ &[J, K_i] = \epsilon_{ij} K_j, \qquad [K_i, H] = P_i. \qquad\qquad (i, j = 1, 2), \end{aligned} \tag{2.3}$$

where ϵ_{ij} is the totally skewsymmetric tensor.

The algebra $\mathcal{NH}$ admits a maximal nontrivial central extension by $\mathbb{R}^3$, associated to the following new non–zero Lie brackets

$$\begin{aligned}
&[P_1, P_2] = \tfrac{1}{\tau^2}F, \qquad [P_1, K_1] = -M, \qquad [H, J] = L,\\
&[K_1, K_2] = F, \qquad [P_2, K_2] = -M.
\end{aligned} \tag{2.4}$$

However, we can neglect the extension associated to L if we realize that it is not a group extension even though it is an algebra extension [14]. This fact relies on the existence of an infinity to one homomorphism relating $SO(2)$ and its universal enveloping group $\mathbb{R}$, ($SO(2) \simeq \mathbb{R}/\mathbb{Z}$). The extended algebra we will consider is therefore eight dimensional.

Since central generators acts trivially on the space–time, the extended group acts on $(t, \mathbf{x})$ in the same way indicated by (2.1). The group law for the extended group $\overline{NH}$can be expressed by

$$\begin{aligned}
\bar{g}'\bar{g} &= (\alpha', \theta', b', \mathbf{a}', \mathbf{v}', \phi')(\alpha, \theta, b, \mathbf{a}, \mathbf{v}, \phi)\\
&\equiv (e^{\alpha' F} e^{\theta' M} e^{b' H} e^{\mathbf{a}'\mathbf{P}} e^{\mathbf{v}'\mathbf{K}} e^{\phi' J})(e^{\alpha F} e^{\theta M} e^{bH} e^{\mathbf{a}\mathbf{P}} e^{\mathbf{v}\mathbf{K}} e^{\phi J})\\
&= \Big(\alpha' + \alpha + \frac{1}{2\tau^2}(\mathbf{a}' \cos\frac{b}{\tau} + \tau\mathbf{v}' \sin\frac{b}{\tau}) \times \mathbf{a}^{\phi'} + \frac{1}{2}(\frac{-\mathbf{a}'}{\tau}\sin\frac{b}{\tau} + \mathbf{v}'\cos\frac{b}{\tau}) \times \mathbf{v}^{\phi'},\\
&\quad \theta' + \theta + \frac{1}{2}(\tau\mathbf{v}'^2 - \frac{\mathbf{a}'^2}{\tau})\sin\frac{b}{\tau}\cos\frac{b}{\tau} - \mathbf{a}'\mathbf{v}'\sin^2\frac{b}{\tau} + \mathbf{v}'\mathbf{a}^{\phi'}\cos\frac{b}{\tau} - \frac{\mathbf{a}'\mathbf{a}^{\phi'}}{\tau}\sin\frac{b}{\tau},\\
&\quad b' + b,\ \mathbf{a}'\cos\frac{b}{\tau} + \mathbf{v}'\tau\sin\frac{b}{\tau} + \mathbf{a}^{\phi'},\ \mathbf{v}'\cos\frac{b}{\tau} - \frac{\mathbf{a}'}{\tau}\sin\frac{b}{\tau} + \mathbf{v}^{\phi'},\ \phi' + \phi\Big).
\end{aligned} \tag{2.5}$$

The inverse of an element g takes the form

$$\begin{aligned}
\bar{g}^{-1} &= (\alpha, \theta, b, \mathbf{a}, \mathbf{v}, \phi)^{-1}\\
&= \Big(-\alpha,\ -\theta - \frac{1}{2}(\tau\mathbf{v}^2 - \frac{\mathbf{a}^2}{\tau})\sin\frac{b}{\tau}\cos\frac{b}{\tau} + \mathbf{a}\mathbf{v}\cos^2\frac{b}{\tau},\\
&\quad -b,\ \big(\tau\mathbf{v}\sin\frac{b}{\tau} - \mathbf{a}\cos\frac{b}{\tau}\big)^{-\phi},\ \big(-\mathbf{v}\cos\frac{b}{\tau} - \frac{\mathbf{a}}{\tau}\sin\frac{b}{\tau}\big)^{-\phi},\ -\phi\Big).
\end{aligned} \tag{2.6}$$

3 Classical elementary systems of $\overline{NH}$

The obtention of classical elementary systems with NH symmetry is achieved by computing the coadjoint orbits of the extended group. The coadjoint

action of a generic element $g=(\alpha,\theta,b,\mathbf{a},\mathbf{v},\phi)\in\overline{NH}$ on a point in $\overline{\mathcal{NH}}^{\,*}$, the dual space of the Lie algebra $\overline{\mathcal{NH}}$, with coordinates $(f,m,h,\mathbf{p},\mathbf{k},j)$ in a dual basis of the basis $\{F,M,H,\mathbf{P},\mathbf{K},J\}$ of $\overline{\mathcal{NH}}$ (and in this order) can be expressed in a compact way as

$$
\begin{aligned}
f' &= f,\\
m' &= m,\\
\mathbf{p}' &= \cos\frac{b}{\tau}\mathbf{p}^{\phi}-\frac{1}{\tau}\sin\frac{b}{\tau}\mathbf{k}^{\phi}-\frac{f}{\tau^2}(\cos\frac{b}{\tau}\mathbf{a}^{\pi/2}-\tau\sin\frac{b}{\tau}\mathbf{v}^{\pi/2})-\frac{m}{\tau}(\tau\cos\frac{b}{\tau}\mathbf{v}+\sin\frac{b}{\tau}\mathbf{a}),\\
\mathbf{k}' &= \tau\sin\frac{b}{\tau}\mathbf{p}^{\phi}+\cos\frac{b}{\tau}\mathbf{k}^{\phi}-\frac{f}{\tau}(\tau\cos\frac{b}{\tau}\mathbf{v}^{\pi/2}+\sin\frac{b}{\tau}\mathbf{a}^{\pi/2})+m(\cos\frac{b}{\tau}\mathbf{a}-\tau\sin\frac{b}{\tau}\mathbf{v}),\\
h' &= h-\mathbf{v}\mathbf{p}^{\phi}+\frac{1}{\tau^2}\mathbf{a}\mathbf{k}^{\phi}+\frac{f}{\tau^2}\mathbf{v}\mathbf{a}^{\pi/2}+\frac{m}{2}(\frac{\mathbf{a}^2}{\tau^2}+\mathbf{v}^2),\\
j' &= j+\mathbf{a}^{\pi/2}\mathbf{p}^{\phi}+\mathbf{v}^{\pi/2}\mathbf{k}^{\phi}-\frac{f}{2}(\frac{\mathbf{a}^2}{\tau^2}+\mathbf{v}^2)-m\mathbf{v}\mathbf{a}^{\pi/2},
\end{aligned}
\tag{3.7}
$$

where $\mathbf{p},\,\mathbf{k}\in\mathbb{R}^2,\;\; h,\,j\in\mathbb{R}$, and we have made use of the following conventions

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{v}\equiv\mathbf{u}\mathbf{v} &= u_1v_1+u_2v_2,\\
|\mathbf{u}|^2\equiv\mathbf{u}^2 &= u_1^2+u_2^2,\\
\mathbf{u}\times\mathbf{v} &= u_1v_2-u_2v_1.
\end{aligned}
\tag{3.8}
$$

This action splits $\overline{\mathcal{NH}}^{\,*}$ into several orbits, which can be classified according to its invariants. We present here a complete classification of the orbits, labeled by the values of the trivial invariants (f,m), which corresponds to the extensions, and the non–trivial invariants (C_i), obtained from (3.7) or by using the kernel of the associated Kirillov two–form on $\overline{\mathcal{NH}}^{\,*}$ [15]. In addition, we give the dimension of the orbits between brackets.

(1) **Non–vanishing extensions** $(f, m \neq 0)$**:**

$$\blacktriangleright \quad (f \neq \pm m\tau) \left\{ \begin{array}{l} C_1 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} - 2mh + \frac{2}{\tau^2} fj \\ C_2 = \mathbf{p} \times \mathbf{k} + mj - fh \end{array} \right\} \quad [4D]. \tag{3.9a}$$

$$\blacktriangleright \quad (f = m\tau) \left\{ \begin{array}{l} 0 < C_3 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} - \frac{2}{\tau}(\mathbf{p} \times \mathbf{k}) \\ C_4 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} + \frac{2}{\tau}(\mathbf{p} \times \mathbf{k}) - \frac{4f}{\tau}(h - \frac{j}{\tau}) \end{array} \right\} \quad [4D]. \tag{3.9b}$$

$$\blacktriangleright \quad (f = -m\tau) \left\{ \begin{array}{l} 0 < C_3' = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} + \frac{2}{\tau}(\mathbf{p} \times \mathbf{k}) \\ C_4' = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} - \frac{2}{\tau}(\mathbf{p} \times \mathbf{k}) + \frac{4f}{\tau}(h + \frac{j}{\tau}) \end{array} \right\} \quad [4D]. \tag{3.9c}$$

$$\blacktriangleright \quad (f = m\tau) \qquad C_3 = 0, \quad C_4, \quad C_5 = h + \frac{j}{\tau}, \quad [2D]. \tag{3.9d}$$

$$\blacktriangleright \quad (f = -m\tau) \qquad C_3' = 0, \quad C_4', \quad C_5' = h - \frac{j}{\tau}, \quad [2D]. \tag{3.9e}$$

(2) **One null extension:**

$$\blacktriangleright \quad (f = 0, m \neq 0) \left\{ \begin{array}{l} C_1 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} - 2mh \\ C_2 = \mathbf{p} \times \mathbf{k} + mj \end{array} \right\} \quad [4D]. \tag{3.9f}$$

$$\blacktriangleright \quad (f \neq 0, m = 0) \left\{ \begin{array}{l} C_1 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} + \frac{2}{\tau^2} fj \\ C_2 = \mathbf{p} \times \mathbf{k} - fh \end{array} \right\} \quad [4D]. \tag{3.9g}$$

(3) **Null extensions** $(f, m = 0)$**:**

► $$(|C_2| < \frac{\tau}{2} C_1) \left\{ \begin{array}{l} 0 < C_1 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} \\ C_2 = \mathbf{p} \times \mathbf{k} \end{array} \right\} \quad [4D]. \tag{3.9h}$$

► $$(C_2 = \frac{\tau}{2} C_1) \left\{ \begin{array}{l} 0 < C_1 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} \\ \mathbf{k} - \tau \mathbf{p}^{\pi/2} = \mathbf{0} \\ C_5 = h + \frac{j}{\tau} \end{array} \right\} \quad [2D]. \tag{3.9i}$$

► $$(C_2 = \frac{-\tau}{2} C_1) \left\{ \begin{array}{l} 0 < C_1 = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} \\ \mathbf{k} + \tau \mathbf{p}^{\pi/2} = \mathbf{0} \\ C_5' = h - \frac{j}{\tau} \end{array} \right\} \quad [2D]. \tag{3.9j}$$

► $$(C_1 = C_2 = 0) \qquad \mathbf{p} = \mathbf{k} = \mathbf{0}, \quad h, \quad j, \quad \text{(points) } [0D]. \tag{3.9k}$$

In the preceding classification we find 4D orbits for non–vanishing values of extensions as well as for null values of one of them or both, although only orbits labeled by (3.9a), (3.9f) and (3.9g) are spaces diffeomorphic to $\mathbb{R}^4$. We must note here the change in the geometry of the orbit when the parameters of extensions are related by the equality $f = \pm m\tau$, a fact that does not occurs for other (2+1) groups (see [8, 16]). Orbits (3.9b) and (3.9c) are diffeomorphic to $\mathbb{R}^3 \times S^1$, while orbits (3.9d) and (3.9e) are to $\mathbb{R}^2$. Concerning the orbits corresponding to vanishing extensions, (3.9h) is diffeomorphic to $\mathbb{R}^2 \times S^1 \times S^1$ while (3.9i) and (3.9j) are to $\mathbb{R} \times S^1$.

We can give a dynamical interpretation of the above systems considering $b(\equiv t)$ as the parameter of time evolution and using the techniques exposed in Ref. [15] for the Galilei group. As an example, we study the dynamics of a system characterized by the pair $(f = 0, m \neq 0)$. Later on, we will see that this restriction is not very significant from a physical point of view when trying to assign a meaning to the central extension defined by the parameter f.

So, let us consider an orbit of type (3.9f) denoted by $O^m_{C_1,C_2}$. A set of canonical coordinates on the orbit (in the sense that their Poisson brackets verify $\{q_i, p_j\} = \delta_{ij},\ \{q_1, q_2\} = \{p_1, p_2\} = 0$) is determined by $\mathbf{q} := \mathbf{k}/m$ and

$\mathbf{p}$. From (3.7) we can write the time evolution of these coordinates as

$$\begin{aligned} \mathbf{q}(t) &= \frac{\mathbf{k}(t)}{m} = \frac{\mathbf{p}(0)}{m}\tau \sin\frac{t}{\tau} + \mathbf{q}(0)\cos\frac{t}{\tau}, \\ \mathbf{p}(t) &= \mathbf{p}(0)\cos\frac{t}{\tau} - \frac{m}{\tau}\mathbf{q}(0)\sin\frac{t}{\tau}. \end{aligned} \tag{3.10}$$

From here we deduce that evolution equations for the system are

$$\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{m}, \qquad \frac{d\mathbf{p}}{dt} = \frac{-m}{\tau^2}\mathbf{q}, \tag{3.11}$$

and therefore, the hamiltonian function is

$$H(\mathbf{q},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \frac{m}{2\tau^2}\mathbf{q}^2 + C, \tag{3.12}$$

which coincides with the value of h obtained from the invariant C_1 of the orbit and represents the energy of a 2D harmonic oscillator with frequency $\omega = 1/\tau$. Invariants C_1 and C_2 are interpreted as the energy and the angular momentum, respectively, and they are constants of motion.

4 Quantum elementary systems of $\overline{NH}$

We have combined both the theory of Mackey of induced representations [17] and the theory of Kirillov for nilpotent groups [18] in order to obtain all the different irreducible representations of $\overline{NH}$. We take advantage of the structure of semi–direct product of a nilpotent subgroup, including extensions, space translations and boosts, times a subgroup containing time translations and rotations.

In this case we can associate in a natural way coadjoint orbits and irreducible representations using an analogous of Kirillov's *orbit method* [19]. We will expose in some detail the procedure of constructing representations associated to the so–called *maximal* orbits (those with $f, m \neq 0$ and $f \neq \pm m\tau$) and summarize the techniques for the rest, giving explicit expressions for all the cases.

$\boxed{\text{3.9a}}$ We choose, among the different decompositions of $\overline{NH}$, the following semidirect product

$$\overline{NH}(2+1) = N \odot K, \tag{4.13}$$

where N is the nilpotent subgroup including central extensions, space translations and boosts, while K is the direct product of the time translations subgroup and rotations subgroup ($K = T \otimes R$).

Firstly, let us calculate a representation for the subgroup N using Kirillov's theory. Coadjoint action is reduced in N to

$$\begin{aligned} f' &= f, \\ m' &= m, \\ \mathbf{p}' &= \mathbf{p} - \frac{f}{\tau^2}\mathbf{a}^{\pi/2} - m\mathbf{v}, \\ \mathbf{k}' &= \mathbf{k} - f\mathbf{v}^{\pi/2} + m\mathbf{a}. \end{aligned} \tag{4.14}$$

The coadjoint orbits lead to different types of irreducible representations of N. We are interested here in those with $f \neq 0$, $m \neq 0$ and $f \neq \pm m\tau$, in order to be able to associate the representations that we will obtain with maximal orbits of the group.

There are two invariants (f, m) of the action (4.14), so the corresponding orbits are diffeomorphic to $\mathbb{R}^4$ and representations are labeled by such constants. Let us choose an arbitrary point $u = (f, m, \mathbf{0}, \mathbf{0})$ on the orbit $O_{f,m}$, and a maximal subalgebra $\mathcal{L}$ of $\mathcal{N}$ subordinate to u, i.e., $\langle u|[B, C]\rangle = 0, \quad \forall\ B, C \in \mathcal{L}$. There are several options, all of them leading to equivalent representations, from which we select the subalgebra spanned by the generators $F, M, P_1 - \frac{1}{\tau}K_1, P_2 + \frac{1}{\tau}K_2$, i.e.,

$$\mathcal{L} = \langle F, M, P_1 - \frac{1}{\tau}K_1, P_2 + \frac{1}{\tau}K_2\rangle. \tag{4.15}$$

It is easy now to find a 1D representation Δ of the group L, obtained via exponentiation of $\mathcal{L}$, by means of duality

$$\Delta(l) = e^{i<u|B>} = e^{i(\xi f + \mu m)}, \tag{4.16}$$

where we use the parametrization

$$l = (\xi, \mu, \delta_1, \delta_2) \ = \ e^B \ = \ e^{\left(\xi F + \mu M + \delta_1 (P_1 - \frac{K_1}{\tau}) + \delta_2 (P_2 + \frac{K_2}{\tau})\right)}. \tag{4.17}$$

The support space of the representation of N is the set of functions defined on the homogeneous space $N/L \equiv X$, $\mathcal{F}(X, \mathbb{C})$. Each coset is characterized by a point $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ in such a way that we define the normalized Borel section

$$\begin{array}{rccl} \sigma : & X & \longrightarrow & N \\ & \mathbf{y} & \mapsto & (0, 0, \mathbf{0}, \mathbf{y}), \end{array} \tag{4.18}$$

with $\pi \circ \sigma = \mathrm{id}_{N/L}$, the canonical projection π being defined by

$$\begin{array}{rccc} \pi: & N & \longrightarrow & X \\ & (\alpha, \theta, \mathbf{a}, \mathbf{v}) & \mapsto & \mathbf{v} - \frac{\mathbf{a}^r}{\tau}, \end{array} \tag{4.19}$$

where $\mathbf{a}^r = (-a_1, a_2)$. This defines the action of N on X through

$$g\mathbf{y} := \pi\big(g\sigma(\mathbf{y})\big) = \mathbf{v} + \mathbf{y} - \frac{\mathbf{a}^r}{\tau}. \tag{4.20}$$

To finish the first step it only remains to induce the representation from L to N, which is done using the fundamental relation

$$[D_{m,f}(g)\varphi](g\mathbf{y}) = \Delta(\sigma^{-1}(g\mathbf{y})g\sigma(\mathbf{y}))\varphi(\mathbf{y}), \qquad \forall g \in N. \tag{4.21}$$

This can be expressed in a suitable form as

$$[D_{m,f}(g)\varphi](\mathbf{y}) = e^{if[\alpha + \frac{1}{2}(\mathbf{v}\times\mathbf{y}) + \frac{1}{2\tau}(\mathbf{v}-\mathbf{y})\times\mathbf{a}^r]}\ e^{im[\theta - \mathbf{y}\mathbf{a} - \frac{1}{2\tau}\mathbf{a}^r\mathbf{a}]}\ \varphi\Big(\mathbf{y} - \mathbf{v} + \frac{\mathbf{a}^r}{\tau}\Big). \tag{4.22}$$

The second step consists of the induction, from the preceding representation, of one for the whole group. For that purpose we must study the action of the subgroup $K = T \otimes R$ on the space of equivalence classes of representations of N, $\hat{N}$, which is parametrized by the values of f and m. This action is carried out by conjugation, i.e.,

$$D'_{m,f}(g) := D_{m,f}(k^{-1}gk), \qquad g \in N,\ k \in K. \tag{4.23}$$

The group K acts trivially on $\hat{N}$ as it is easy to see, and therefore, the little group $L_{m,f}$ coincides with K. Consequently, the representations of $N \odot L_{m,f}$, from which we achieve the induction, are truly representations of $\overline{NH}$. More explicitly, we use the fact that $L_{m,f}$ is a direct product to accomplish induction first with time translations and then with rotations. To start with, we construct a representation D^t of $N \odot T$ by

$$D^t(g) := D(t^{-1}gt), \quad g \in N,\ t \in T, \tag{4.24}$$

giving raise to

$$
\begin{aligned}
[D^t_{m,f}(g)\varphi](\mathbf{y}) = \exp\{if[\alpha + \frac{1}{2}(\mathbf{v}\cos\frac{b}{\tau} - \frac{\mathbf{a}}{\tau}\sin\frac{b}{\tau}) \times \mathbf{y} \\
+ \frac{1}{2\tau}((\mathbf{v}\cos\frac{b}{\tau} - \frac{\mathbf{a}}{\tau}\sin\frac{b}{\tau}) - \mathbf{y}) \times (\mathbf{a}\cos\frac{b}{\tau} + \mathbf{v}\tau\sin\frac{b}{\tau})^r]\} \\
\times \exp\{im[\theta + \frac{1}{2}(\tau\mathbf{v}^2 - \frac{\mathbf{a}^2}{\tau})\sin\frac{b}{\tau}\cos\frac{b}{\tau} - \mathbf{av}\sin^2\frac{b}{\tau} - \mathbf{y}(\mathbf{a}\cos\frac{b}{\tau} + \mathbf{v}\tau\sin\frac{b}{\tau}) \\
- \frac{1}{2\tau}(\mathbf{a}\cos\frac{b}{\tau} + \mathbf{v}\tau\sin\frac{b}{\tau})^r(\mathbf{a}\cos\frac{b}{\tau} + \mathbf{v}\tau\sin\frac{b}{\tau})]\} \\
\times \varphi\Big(\mathbf{y} - \mathbf{v}\cos\frac{b}{\tau} + \frac{\mathbf{a}}{\tau}\sin\frac{b}{\tau} + \frac{1}{\tau}\big(\mathbf{a}\cos\frac{b}{\tau} + \mathbf{v}\tau\sin\frac{b}{\tau}\big)^r\Big). \quad (4.25)
\end{aligned}
$$

The following step is to check the existence of an operator $W(b)$ realizing the equivalence in the sense of Mackey's theory, i.e., verifying

$$
D^t_{m,f}(g) = W^{-1}(b)\; D_{m,f}(g)\; W(b), \qquad \forall g \in N,\ \forall t \in T. \tag{4.26}
$$

This operator is reduced in most cases to exponentials of Casimir operators, which formally coincides with the invariants of the coadjoint action for the considered group $(N \odot T)$. For the whole group $\overline{NH}$ such invariants are

$$
\begin{aligned}
C_1 &= \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} - 2mh + \frac{2}{\tau^2}fj, \\
C_2 &= \mathbf{p} \times \mathbf{k} + mj - fh.
\end{aligned} \tag{4.27}
$$

Omitting rotations, this two invariants are reduced to one in order to maintain the even dimension of the orbit and this new one can be obtained eliminating j in the preceding equations. It turns out to be $C = C_1 - \frac{2f}{\tau^2 m}C_2$, i.e.,

$$
C = \mathbf{p}^2 + \frac{\mathbf{k}^2}{\tau^2} + \Big(\frac{2f^2}{\tau^2 m} - 2m\Big)h - \frac{2f}{\tau^2 m}(\mathbf{p} \times \mathbf{k}). \tag{4.28}
$$

If we solve for h in this last expression and reinterpret it in terms of the algebra generators we get

$$
\hat{H} = \frac{\tau^2 m}{2(\tau^2 m^2 - f^2)}\,\Big(\hat{\mathbf{P}}^2 + \frac{\hat{\mathbf{K}}^2}{\tau^2} - \frac{2f}{\tau^2 m}(\hat{\mathbf{P}} \times \hat{\mathbf{K}}) - C\hat{I}\Big), \tag{4.29}
$$

where we have consider $\hat{M} = m$ and $\hat{F} = f$ since we are in the orbit $O_{f,m}$, while $\hat{\mathbf{P}}$ and $\hat{\mathbf{K}}$ can be calculated from (4.22) and whose differential expressions are

$$\begin{aligned} \hat{\mathbf{P}} &= \frac{f}{2\tau}(\mathbf{y}^{\frac{-\pi}{2}})^r - m\mathbf{y} - \frac{i}{\tau}\nabla^R_{\mathbf{y}}, \\ \hat{\mathbf{K}} &= \frac{f}{2}\mathbf{y}^{\frac{-\pi}{2}} + i\nabla_{\mathbf{y}}. \end{aligned} \tag{4.30}$$

The intertwining operator verifying identity (4.26) can be defined by $W(b) = e^{i\hat{H}b}$. Thus, we get the induced representation $U_{m,f,C}$ for $N \odot T$ given by

$$\begin{aligned} [U_{m,f,C}(g)\varphi](\mathbf{y}) &= e^{if[\alpha+\frac{1}{2}(\mathbf{v}\times\mathbf{y})+\frac{1}{2\tau}(\mathbf{v}-\mathbf{y})\times\mathbf{a}^r]}\ e^{im[\theta-\mathbf{y}\mathbf{a}-\frac{1}{2\tau}\mathbf{a}^r\mathbf{a}]} \\ &\quad\times e^{i\frac{\tau^2 m}{2(\tau^2m^2-f^2)}(\hat{\mathbf{P}}^2+\frac{\hat{\mathbf{K}}^2}{\tau^2}-\frac{2f}{\tau^2 m}(\hat{\mathbf{P}}\times\hat{\mathbf{K}})-C)b}\ \varphi\Big(\mathbf{y}-\mathbf{v}+\frac{\mathbf{a}^r}{\tau}\Big) \end{aligned} \tag{4.31}$$

where we have chosen a 1D irreducible unitary representation (i.u.r.) of T ($b \in T \to e^{ibn},\ n \in \mathbb{Z}$) which is included in constant C.

Following the same method we can complete the induction process and construct the i.u.r. for the whole group $\overline{NH}$ departing from the ones just calculated. Since $N \odot T$ is not abelian, we conjugate a generic element $g \in N \odot T$ with a rotation $R_\phi \in R$, obtaining

$$g^\phi = R^{-1}(\phi)gR(\phi) = (\alpha, \theta, b, \mathbf{a}^{-\phi}, \mathbf{v}^{-\phi}, 0). \tag{4.32}$$

From here we construct the following representation of $N \odot T$

$$\begin{aligned} [U^\phi_{m,f\,C}(g)\varphi](\mathbf{y}) &= e^{if[\alpha+\frac{1}{2}(\mathbf{v}^{-\phi}\times\mathbf{y})+\frac{1}{2\tau}(\mathbf{v}^{-\phi}-\mathbf{y})\times(\mathbf{a}^{-\phi})^r]}\ e^{im[\theta-\mathbf{y}\mathbf{a}^{-\phi}-\frac{1}{2\tau}(\mathbf{a}^{-\phi})^r\mathbf{a}^{-\phi}]} \\ &\quad\times e^{i\frac{\tau^2 m}{2(\tau^2m^2-f^2)}[\hat{\mathbf{P}}^2+\frac{\hat{\mathbf{K}}^2}{\tau^2}-\frac{2f}{\tau^2 m}(\hat{\mathbf{P}}\times\hat{\mathbf{K}})-C]b} \\ &\quad\times \varphi\Big(\mathbf{y}-\mathbf{v}^{-\phi}+\frac{(\mathbf{a}^{-\phi})^r}{\tau}\Big). \end{aligned} \tag{4.33}$$

Now, since rotations acts trivially on time translations (remember that $K = T \otimes R$), we can neglect the latter and consider $\tilde{g} \in NH/T$, with a considerable saving of calculation. We must find again an intertwining operator $W(\phi)$ such that

$$U^\phi(\tilde{g}) = W^{-1}(\phi)\ U(\tilde{g})\ W(\phi). \tag{4.34}$$

We take $W(\phi) = e^{i\phi \hat{J}}$, where $\hat{J}$ is obtained on solving for j in the expression (4.27) of the invariant C_2, i.e.,

$$\hat{J} = \frac{1}{m}(C_2 - \hat{\mathbf{P}} \times \hat{\mathbf{K}} + f\hat{H}), \qquad W(\phi) = e^{\frac{-i}{m}(\hat{\mathbf{P}}\times\hat{\mathbf{K}} - f\hat{H} - C_2)\phi}. \tag{4.35}$$

Finally, the expression for the representation associated to the orbit labeled by $O_{f,m}^{C_1,C_2}$ turns out to be

$$\begin{aligned}[U_{m,f}^{C_1,C_2}(g)\varphi](\mathbf{y}) = {} & e^{if[\alpha+\frac{1}{2}(\mathbf{v}\times\mathbf{y})+\frac{1}{2\tau}(\mathbf{v}-\mathbf{y})\times\mathbf{a}^r]}\ e^{im[\theta-\mathbf{ya}-\frac{1}{2\tau}\mathbf{a}^r\mathbf{a}]} \\ & \times e^{i\frac{\tau^2 m}{2(\tau^2m^2-f^2)}[\hat{\mathbf{P}}^2+\frac{\hat{\mathbf{K}}^2}{\tau^2}-\frac{2f}{\tau^2 m}(\hat{\mathbf{P}}\times\hat{\mathbf{K}})-C_1+\frac{2f}{\tau^2 m}C_2]b} \\ & \times e^{\frac{-i}{m}(\hat{\mathbf{P}}\times\hat{\mathbf{K}}-f\hat{H}-C_2)\phi}\ \varphi\Big(\mathbf{y}-\mathbf{v}+\frac{\mathbf{a}^r}{\tau}\Big),\end{aligned} \tag{4.36}$$

where the constants C_2 and C_1 in the exponentials include, respectively, an integer number n' corresponding to the 1D i.u.r. $e^{in'\phi}$ of $SO(2)$ ($\phi \to e^{in'\phi},\ n' \in \mathbb{Z}$) and a real number associated to the i.u.r. of T before mentioned.

$\boxed{3.9\text{b}}$ When we consider $f = m\tau$, the coadjoint orbits for the nilpotent factor N are characterized by the invariants f and $\mathbf{x} = \mathbf{p}+\frac{\mathbf{k}^{\pi/2}}{\tau}$, so every i.u.r. of N is labeled by the values $(f, \mathbf{x})$. In this case, the maximal subalgebra subordinate to a point on the orbit must be 4D, for example, we take

$$\mathcal{L} = \langle F,\ P_1 - \frac{1}{\tau}K_1,\ P_2 + \frac{1}{\tau}K_2,\ P_1 - \frac{1}{\tau}K_2\rangle. \tag{4.37}$$

This leads to the following representation for N

$$\begin{aligned}[D_{f,\mathbf{x}}(n)\varphi](y) = {} & e^{if[\alpha+y(v_1-\frac{a_2}{\tau})-\frac{1}{2}(v_1^2+v_1v_2+\frac{a_2^2-a_1a_2}{\tau^2})+\frac{v_1}{\tau}(a_2-a_1)]} \\ & \times e^{i\mathbf{xa}}\ \varphi\big(y - v_1 - v_2 - \frac{a_1}{\tau} + \frac{a_2}{\tau}\big),\end{aligned} \tag{4.38}$$

defined on $\mathcal{L}^2(\mathbb{R})$.

The action of K on $\hat{N}$ is given by $(f', \mathbf{x}') = (f, \mathbf{x}^{(\frac{b}{\tau}+\phi)})$, so there are two kinds of strata depending on the value of the invariant $\mathbf{x}^2$. In this case we choose $\mathbf{x}^2 \neq 0$ in order to associate this value with that of constant C_3 defining the strata of orbits we are dealing with (the case $C_3 = 0$ will be studied later on for orbits (3.9d)).

The little group of the orbits of the strata characterized by $\mathbf{x}^2 \neq 0$ is $L_- = \{(b, \phi) \in T \otimes R \mid \frac{b}{\tau} + \phi = 0\}$, which is generated by $H - \frac{1}{\tau}J$. The

operator $W(l_-)$ carrying out the equivalence between representation $D_{f,\mathbf{x}}$ and its conjugate by the action of K, $D^k_{f,\mathbf{x}}$, is

$$W\big(b, \frac{-b}{\tau}\big) = e^{ib(\widehat{H - \frac{1}{\tau}J})} = e^{ib\frac{\tau}{4f}(\hat{\mathbf{P}}^2 + \frac{\hat{\mathbf{K}}^2}{\tau^2} + \frac{2}{\tau}(\hat{\mathbf{P}}\times\hat{\mathbf{K}}) - C_4\hat{I})}, \tag{4.39}$$

where the differential representation of $\hat{\mathbf{P}}$ and $\hat{\mathbf{K}}$ are obtained from (4.38). Thus, the representation Υ of $N \odot L_-$ from which to induce is the tensor product of the representation e^{ibC} of L_- times $D_{f,\mathbf{x}}(n)W(l_-)$ of N.

The homogeneous space $\overline{NH}/(N\odot L_-)$ is isomorphic to $\mathbb{R}$ and its elements will be denoted by t, with canonical projection $\pi : \overline{NH} \longrightarrow \overline{NH}/(N \odot L_-)$ given by $\pi(\alpha, b, \mathbf{a}, \mathbf{v}, \phi) = \frac{b}{\tau} + \phi$ and normalized Borel section defined by $s(t) = (0, \tau t, \mathbf{0}, \mathbf{0}, 0)$.

The i.u.r. of the group $\overline{NH}$ induced by the representation Υ of $N \odot L_-$ is expressed by

$$[U_f^{C_3,C_4}\Psi](t) = \Upsilon(\eta)\ \Psi(t - \frac{b}{\tau} - \phi) \tag{4.40}$$

where $\Psi \in \mathcal{L}^2(\mathbb{R})$, and $\eta = s^{-1}(t)gs(g^{-1}t)$, explicitly

$$\begin{aligned}\eta = \Big(&\alpha + \frac{1}{2}\cos(t - \frac{b}{\tau} - \phi)\sin(t - \frac{b}{\tau} - \phi)(\mathbf{v}^2 - \frac{\mathbf{a}^2}{\tau^2}) - \frac{\mathbf{a}\mathbf{v}}{\tau}\sin^2(t - \frac{b}{\tau} - \phi),\\ &-\tau\phi,\ \mathbf{a}\cos(t - \frac{b}{\tau} - \phi) + \tau\mathbf{v}\sin(t - \frac{b}{\tau} - \phi),\\ &\mathbf{v}\cos(t - \frac{b}{\tau} - \phi) - \frac{\mathbf{a}}{\tau}\sin(t - \frac{b}{\tau} - \phi),\ \phi\Big).\end{aligned} \tag{4.41}$$

$\boxed{3.9c}$ This case is analogous to the preceding one. Invariants characterizing the orbits for the nilpotent part are $(f,\ \mathbf{y} = \mathbf{p} - \frac{\mathbf{k}^{\pi/2}}{\tau})$ and we can choose the subalgebra

$$\mathcal{L} = \langle F,\ P_1 - \frac{1}{\tau}K_1,\ P_2 + \frac{1}{\tau}K_2,\ P_1 + \frac{1}{\tau}K_2\rangle \tag{4.42}$$

in order to obtain the following i.u.r. for N defined on $\mathcal{L}^2(\mathbb{R})$

$$\begin{aligned}[D_{f,\mathbf{y}}(n)\varphi](x) = e^{if[\alpha - x(v_1 + \frac{a_2}{\tau}) + \frac{1}{2}(v_1^2 - v_1v_2 + \frac{a_2^2 + a_1a_2}{\tau^2}) + \frac{v_1}{\tau}(a_1 + a_2)]}&\\ \times e^{i\mathbf{y}\mathbf{a}}\ \varphi(x - v_1 + v_2 - \frac{1}{\tau}(a_1 + a_2)).&\end{aligned} \tag{4.43}$$

The action of K on $\hat{N}$ is $(f', \mathbf{y}') = (f, \mathbf{y}^{(\frac{b}{\tau}-\phi)})$. The little group of the orbit characterized by $\mathbf{y}^2 = C_3' \neq 0$ is generated by $H + \frac{1}{\tau}J$, i.e.,

$$L_+ = \{(b,\phi) \in T \otimes R \mid \frac{b}{\tau} - \phi = 0\}. \tag{4.44}$$

The operator $W(l_+)$ relating representations $D_{f,\mathbf{y}}$ and $D^k_{f,\mathbf{y}},\ k \in K$ is

$$W(b, \frac{b}{\tau}) = e^{ib(\widehat{H+\frac{1}{\tau}J})} = e^{-ib\frac{\tau}{4f}(\hat{\mathbf{P}}^2 + \frac{\hat{\mathbf{K}}^2}{\tau^2} - \frac{2}{\tau}(\hat{\mathbf{P}}\times\hat{\mathbf{K}}) - C_4'\hat{I})}, \tag{4.45}$$

where the values of $\hat{\mathbf{P}}$ and $\hat{\mathbf{K}}$ are implicit in (4.43). The representation Υ of $N \odot L_+$ from which we induce is the tensor product of the representation of L_+, i.e., $(e^{ibC'})$ times $D_{f,\mathbf{y}}(n)W(l_+)$.

Finally, the induced representation is

$$[U_f^{C_3',C_4'}\Psi](t) = \Upsilon(\eta')\ \Psi\big(t - \frac{b}{\tau} + \phi\big) \tag{4.46}$$

where $\Psi \in \mathcal{L}^2(\mathbb{R})$ and the element of induction, η, is

$$\begin{aligned}\eta' = \Big(&\alpha + \frac{1}{2}\cos(t-\frac{b}{\tau}+\phi)\sin(t-\frac{b}{\tau}+\phi)(\mathbf{v}^2 - \frac{\mathbf{a}^2}{\tau^2}) - \frac{\mathbf{a}\mathbf{v}}{\tau}\sin^2(t-\frac{b}{\tau}+\phi),\\ &\tau\phi,\ \mathbf{a}\cos(t-\frac{b}{\tau}+\phi) + \tau\mathbf{v}\sin(t-\frac{b}{\tau}+\phi),\\ &\mathbf{v}\cos(t-\frac{b}{\tau}+\phi) - \frac{\mathbf{a}}{\tau}\sin(t-\frac{b}{\tau}+\phi),\ \phi\Big).\end{aligned} \tag{4.47}$$

$\boxed{\text{3.9d}}$ This case completes the study started in 3.9b since corresponds to take $C_3 \equiv \mathbf{x}^2 = 0$. The action of K on $\hat{N}$ is trivial and the little group coincides with K. Therefore, the representations from which we realize the induction are true representations of $\overline{NH}$ and it is not necessary to take the last step of case 3.9b. Consequently, we must consider an i.u.r. of the abelian group K, $\rho_{\kappa_1,\kappa_2}(b,\phi) = e^{i(b\kappa_1+\phi\kappa_2)}$, the representation of the nilpotent part $(D_{f,\mathbf{x}=\mathbf{0}})$ and the intertwining operator $W(b,\phi)$ realizing the equivalence in the sense of Mackey. This operator can be written in terms of the generators $H+\frac{1}{\tau}J$ and $H-\frac{1}{\tau}J$, which can be directly obtained from the coadjoint action invariants C_4 and C_5, explicitly

$$W(b,\phi) = e^{\frac{i}{2}(b+\tau\phi)C_5}\ e^{i\frac{\tau}{8f}(b-\tau\phi)[\hat{\mathbf{P}}^2 + \frac{\hat{\mathbf{K}}^2}{\tau^2} + \frac{2}{\tau}(\hat{\mathbf{P}}\times\hat{\mathbf{K}}) - C_4\hat{I}]}. \tag{4.48}$$

The final expression for the i.u.r. is

$$U_f^{C_4,C_5}(g) = \rho_{\kappa_1,\kappa_2}(k) \otimes D_{f,\mathbf{x}=\mathbf{0}}(n) W(b,\phi). \tag{4.49}$$

$\boxed{3.9\text{e}}$ This case can be solved as the preceding one and complete the case 3.9c since now we consider $C_3' \equiv \mathbf{y}^2 = 0$. The little group of the action of K on $\hat{N}$ is again $T \otimes R$. Hence, the corresponding i.u.r. is

$$U_f^{C_4',C_5'}(g) = \rho_{\kappa_1,\kappa_2}(k) \otimes D_{f,\mathbf{y}=\mathbf{0}}(n) W(b,\phi), \tag{4.50}$$

where ρ is the same character of previous case and the operator W is now

$$W(b,\phi) = e^{\frac{i}{2}(b-\tau\phi)C_5'}\; e^{i\frac{\tau}{8f}(b+\tau\phi)[\hat{\mathbf{P}}^2+\frac{\hat{\mathbf{K}}^2}{\tau^2}-\frac{2}{\tau}(\hat{\mathbf{P}}\times\hat{\mathbf{K}})-C_4'\hat{I}]}. \tag{4.51}$$

$\boxed{3.9\text{f}}$ The construction of this representation follows the procedure used for maximal orbits. In this case, there is only one invariant, m, for the coadjoint action of the nilpotent factor of the group, so the coadjoint orbits are isomorphic to $\mathbb{R}^4$ and representations are labeled by this constant.

On the other hand, condition $f = 0$ is equivalent to the fact that generators of space translations commute, i.e., $[P_1, P_2] = 0$. This allows us to choose the maximal subordinate subalgebra to a point in the orbit as $\mathcal{L} =< M, P_1, P_2 >$, which facilitates calculations and leads to a representation equivalent to the obtained by using subalgebra (4.15), which would have an expression similar to that found in (4.36).

The i.u.r. is defined on $\mathcal{L}^2(\mathbb{R}^2)$, with velocity kind argument and takes the form

$$[U_m^{C_1,C_2}(g)\varphi](\mathbf{y}) = e^{im[\theta-\mathbf{y}\mathbf{a}]}\; e^{\frac{i}{2m}(\hat{\mathbf{P}}^2+\frac{\hat{\mathbf{K}}^2}{\tau^2}-C_1\hat{I})b}\; e^{\frac{-i}{m}(\hat{\mathbf{P}}\times\hat{\mathbf{K}}-C_2\hat{I})\phi}\; \varphi(\mathbf{y}-\mathbf{v}), \tag{4.52}$$

where constants C_1, C_2 include the parameters labeling the i.u.r. of T and rotations. In section 5.1 we use the differential form of operators $\hat{\mathbf{P}}, \hat{\mathbf{K}}$ acting on functions in $\mathcal{L}^2(\mathbb{R}^2)$ as $\hat{\mathbf{P}} = -m\mathbf{y}$, $\hat{\mathbf{K}} = i\nabla_{\mathbf{y}}$ in order to construct the SW correspondence.

$\boxed{3.9\text{g}}$ The process is completely analogous to the preceding one interchanging the roles played by m and f. There are only two differences, the first one is the choice of the subalgebra $\mathcal{L} = \langle F, P_1 - \frac{1}{\tau}K_1, P_2 + \frac{1}{\tau}K_2\rangle$, the same as in the maximal case. The second one appears as a consequence of $m = 0$ in the orbit invariants when defining the operators $\hat{H}$ and $\hat{J}$, although

the option is obvious and can be noticed in the final result, which again is a representation acting on $\mathcal{L}^2(\mathbb{R}^2)$

$$
\begin{aligned}
[U_f^{C_1,C_2}(g)\varphi](\mathbf{y}) = e^{if[\alpha+\frac{1}{2}(\mathbf{v}\times\mathbf{y})+\frac{1}{2\tau}(\mathbf{v}-\mathbf{y})\times\mathbf{a}^r]}\; e^{i\frac{1}{f}(\hat{\mathbf{P}}\times\hat{\mathbf{K}}-C_2)b} \\
\times\, e^{i\frac{\tau^2}{2f}(\hat{\mathbf{P}}^2+\frac{\hat{\mathbf{K}}^2}{\tau^2}-C_1)\phi}\; \varphi\Big(\mathbf{y}-\mathbf{v}+\frac{\mathbf{a}^r}{\tau}\Big),
\end{aligned}
\tag{4.53}
$$

where C_1 and C_2 include the labels of the i.u.r. of one–parameter groups as in the previous cases.

$\boxed{\text{3.9h}}$ The remaining cases correspond to the unextended group, i.e., $f = m = 0$.

The i.u.r. of the abelian kernel N (space translations and boosts) are labeled by the characters. These 1D representations are defined by

$$
D_{\boldsymbol{\rho},\boldsymbol{\varkappa}}(\mathbf{a},\mathbf{v}) := e^{i(\boldsymbol{\rho}\mathbf{a}+\boldsymbol{\varkappa}\mathbf{v})}, \qquad \boldsymbol{\rho},\, \boldsymbol{\varkappa} \in \mathbb{R}^2. \tag{4.54}
$$

The action of the other group factor, K, on the set of characters $(\boldsymbol{\rho},\boldsymbol{\varkappa})$ is easily computed and results to be

$$
\begin{aligned}
\boldsymbol{\rho}' &= \boldsymbol{\rho}^{\phi}\cos\frac{b}{\tau} - \frac{1}{\tau}\boldsymbol{\varkappa}^{\phi}\sin b\tau, \\
\boldsymbol{\varkappa}' &= \tau\boldsymbol{\rho}^{\phi}\sin\frac{b}{\tau} + \boldsymbol{\varkappa}^{\phi}\cos b\tau.
\end{aligned}
\tag{4.55}
$$

This action is formally identical to the restriction to K of the coadjoint action (3.7) for $(\mathbf{p},\mathbf{k})$ and, therefore, its invariants are either identical. So we can write

$$
\begin{aligned}
\boldsymbol{\rho}^2 + \frac{\boldsymbol{\varkappa}^2}{\tau^2} &= I_1, \\
\boldsymbol{\rho}\times\boldsymbol{\varkappa} &= I_2.
\end{aligned}
\tag{4.56}
$$

Depending on the values assigned to I_1 and I_2, we have some of the possible four kinds of orbits and hence, of representations.

As first case, we suppose $I_2 \neq \pm\frac{\tau}{2}I_1$, which results in 2D character orbits, whose little group is trivial. The induced representations act on functions defined on the homogeneous space $NH/N \backsimeq \mathbb{R}^2$, where we take the section $s(\mathbf{t}) = (t_1, \mathbf{0}, \mathbf{0}, t_2)$ with projection $\pi(b,\mathbf{a},\mathbf{v},\phi) = (b,\phi)$. The action of the

group NH on this space is $g\mathbf{t} = (t_1 + b, t_2 + \phi)$ and the induction element is

$$
\begin{aligned}
s^{-1}(\mathbf{t})\, g\, s(g^{-1}\mathbf{t}) &= (-t_1, \mathbf{0}, \mathbf{0}, -t_2)(b, \mathbf{a}, \mathbf{v}, \phi)(t_1 - b, \mathbf{0}, \mathbf{0}, t_2 - \phi) \\
&= (0,\, \mathbf{a}^{-t_2} \cos \frac{t_1 - b}{\tau} + \tau \mathbf{v}^{-t_2} \sin \frac{t_1 - b}{\tau}, \\
&\qquad \mathbf{v}^{-t_2} \cos \frac{t_1 - b}{\tau} - \frac{1}{\tau} \mathbf{a}^{-t_2} \sin \frac{t_1 - b}{\tau},\, 0).
\end{aligned}
\tag{4.57}
$$

The induced i.u.r. is expressed as

$$
\begin{aligned}
[U_{C_1,C_2}(g)\varphi](\mathbf{t}) &= e^{i\boldsymbol{\rho}(\mathbf{a}^{-t_2} \cos \frac{t_1-b}{\tau} + \tau \mathbf{v}^{-t_2} \sin \frac{t_1-b}{\tau})} \\
&\quad \times e^{i\boldsymbol{\varkappa}(\mathbf{v}^{-t_2} \cos \frac{t_1-b}{\tau} - \frac{1}{\tau}\mathbf{a}^{-t_2} \sin \frac{t_1-b}{\tau})}\, \varphi(t_1 - b, t_2 - \phi),
\end{aligned}
\tag{4.58}
$$

with $C_1 = \boldsymbol{\rho}^2 + \frac{\boldsymbol{\varkappa}^2}{\tau^2}$ and $C_2 = \boldsymbol{\rho} \times \boldsymbol{\varkappa}$.

$\boxed{3.9\text{i}}$ Departing from the preceding case and taking $I_2 = (\tau/2)I_1$, we get the new invariant $\boldsymbol{\rho} + \frac{\boldsymbol{\varkappa}^{\pi/2}}{\tau} = \mathbf{0}$ and conclude that the action on the characters is reduced to the rotation

$$
\boldsymbol{\varkappa}' = \boldsymbol{\varkappa}^{(\frac{b}{\tau} - \phi)},
\tag{4.59}
$$

and hence, the corresponding little group, generated by $H + \frac{1}{\tau}J$, is

$$
L_+ = \{(b, \phi) \in T \otimes R \mid \frac{b}{\tau} - \phi = 0\}.
\tag{4.60}
$$

The i.u.r. of L_+ are $\Delta_C(l) = e^{ibC}$, where b is the parameter corresponding to the element of L_+ written as $(b,\, b/\tau)$. The real constant C can be associated to the value of constant C_5 of the coadjoint orbit giving the value of the invariant $h + j/\tau$.

Therefore, we induce from the representation of $N \odot L_+$, $U(n, h_+) = e^{ibC_5}\, e^{i(\boldsymbol{\rho}\mathbf{a} + \boldsymbol{\varkappa}\mathbf{v})}$. The homogeneous space $NH/(N \odot L_+)$ is now isomorphic to $\mathbb{R}$, and we can define the section $s(t) = (\tau t, \mathbf{0}, \mathbf{0}, 0)$ with canonical projection $\pi(b, \mathbf{a}, \mathbf{v}, \phi) = \frac{b}{\tau} - \phi$. The induction element is

$$
\begin{aligned}
s^{-1}(t)\, g\, s(g^{-1}t) &= (\tau\phi,\, \mathbf{a} \cos(t - \frac{b}{\tau} + \phi) + \tau \mathbf{v} \sin(t - \frac{b}{\tau} + \phi), \\
&\qquad \mathbf{v} \cos(t - \frac{b}{\tau} + \phi) - \frac{1}{\tau}\mathbf{a} \sin(t - \frac{b}{\tau} + \phi),\, \phi),
\end{aligned}
\tag{4.61}
$$

and the representation is

$$[U_{C_1,C_5}(g)\varphi](t) = e^{i\tau\phi C_5}\ e^{i\boldsymbol{\rho}[\mathbf{a}\cos(t-\frac{b}{\tau}+\phi)+\tau\mathbf{v}\sin(t-\frac{b}{\tau}+\phi)]} \times e^{i\boldsymbol{\varkappa}[\mathbf{v}\cos(t-\frac{b}{\tau}+\phi)-\frac{1}{\tau}\mathbf{a}\sin(t-\frac{b}{\tau}+\phi)]}\ \varphi\big(t-\frac{b}{\tau}+\phi\big), \tag{4.62}$$

with $C_1 = \boldsymbol{\rho}^2 + \frac{\boldsymbol{\varkappa}^2}{\tau^2}$ and $\boldsymbol{\rho} + \frac{\boldsymbol{\varkappa}^{\pi/2}}{\tau} = \mathbf{0}$.

$\boxed{\text{3.9j}}$ Construction process is identical to the previous case, so we do not repeat it here. We only have to notice that the little group is now generated by $H - \frac{1}{\tau}J$, leading to the i.u.r.

$$[U_{C_1,C_5'}(g)\varphi](t) = e^{-i\tau\phi C_5'}\ e^{i\boldsymbol{\rho}[\mathbf{a}\cos(t-\frac{b}{\tau}-\phi)+\tau\mathbf{v}\sin(t-\frac{b}{\tau}-\phi)]} \times e^{i\boldsymbol{\varkappa}[\mathbf{v}\cos(t-\frac{b}{\tau}-\phi)-\frac{1}{\tau}\mathbf{a}\sin(t-\frac{b}{\tau}-\phi)]}\ \varphi\big(t-\frac{b}{\tau}-\phi\big), \tag{4.63}$$

with $C_1 = \boldsymbol{\rho}^2 + \frac{\boldsymbol{\varkappa}^2}{\tau^2}$ and $\boldsymbol{\rho} - \frac{\boldsymbol{\varkappa}^{\pi/2}}{\tau} = \mathbf{0}$.

$\boxed{\text{3.9k}}$ This last case corresponds to the identity character ($\boldsymbol{\rho} = \boldsymbol{\varkappa} = 0$). So, the little group is $T \otimes R$ and, therefore, the i.u.r. is reduced to the character of this abelian group, i.e.,

$$U_{h,j}(g) = e^{ibh}\ e^{i\phi j}. \tag{4.64}$$

5 An example of Moyal Quantization

Quantization of elementary systems has been a subject of interest in Physics as well as in Mathematics during last decades. It is due to the fact that it seeks to answer the open questions about the deep meaning of quantum theory and its relation with the classical one, as well as it tries to find a mathematical structure able to globally formalize quantum phenomena. We can mention different approaches to the problem, such as geometric quantization [20, 21], quantization on groups [22, 23], quantization by deformation [24] or Fedosov's quantization [25]. All of them are conditioned by the formalism that describe quantum systems. In this sense, we consider here the so–called Moyal quantization, which is based on the Moyal formulation of Quantum Mechanics or phase space formalism [26], whose main ingredient is the consideration of both observables and states as (generalized) functions on a given phase space, introducing quantum concepts by means of an associative but non commutative product called "twisted product".

In particular, we use the Stratonovich–Weyl (SW) correspondence, which departing from the ideas of Stratonovich [27] has been developed during last years for different kinds of systems [28, 29, 3, 4, 5]. We refer the reader to the last three references for more details or to [30, 31] for a review paper.

As an example of how this quantization scheme works for the classical elementary systems that we described in section 3, we accomplish in next subsection the quantization of orbits of kind 3.9f.

5.1 Example of quantization

Let us consider one orbit of type 3.9f labeled by the three invariants m, C_1 and C_2, $O^m_{C_1,C_2}$. We already saw that ($\mathbf{q} = \mathbf{k}/m$, $\mathbf{p}$) constitute a set of canonical coordinates ($\{q_i, p_j\} = \delta_{ij}$, $\{q_1, q_2\} = \{p_1, p_2\} = 0$). Moreover, we can evaluate h and j departing from the values of $\mathbf{q}$, $\mathbf{p}$, m, C_1 and C_2 obtaining

$$h = \frac{\mathbf{p}^2}{2m} + \frac{m\mathbf{q}^2}{2\tau^2} - \frac{C_1}{2m}, \qquad j = \frac{C_2}{m} - \mathbf{p} \times \mathbf{q}. \tag{5.65}$$

Thus, the orbit is isomorphic to $\mathbb{R}^4$.

We can fix as origin in the orbit the point ($\mathbf{q} = \mathbf{0}$, $\mathbf{p} = \mathbf{0}$) and define on it the autoadjoint parity operator

$$[\Omega(\mathbf{0}, \mathbf{0})\varphi](\mathbf{y}) := 2^2\, \varphi(-\mathbf{y}) \tag{5.66}$$

(see Ref. [3] for more details). This definition of $\Omega(\mathbf{0}, \mathbf{0})$ is an Ansatz for the SW kernel that we must extend to the rest of the orbit using covariance property (this kernel is the central object of the SW correspondence, that relates functions on the considered phase space with operators on a suitable Hilbert space), i.e.,

$$\Omega(\mathbf{q}, \mathbf{p}) \equiv \Omega\big(g(\mathbf{0}, \mathbf{0})\big) = U(g)\Omega(\mathbf{0}, \mathbf{0})U(g^{-1}), \tag{5.67}$$

where $g(\mathbf{0}, \mathbf{0}) = (\mathbf{q}, \mathbf{p})$. One such element g of $\overline{NH}$ is

$$g_{\mathbf{qp}} = (0, 0, \mathbf{q}, -\mathbf{p}/m, 0). \tag{5.68}$$

We get the SW kernel at any point $(\mathbf{q}, \mathbf{p})$ as

$$[\Omega(\mathbf{q}, \mathbf{p})\varphi](\mathbf{y}) = 2^2\, e^{-2i\mathbf{q}(m\mathbf{y}+\mathbf{p})}\, \varphi(-\mathbf{y} - 2\frac{\mathbf{p}}{m}). \tag{5.69}$$

This definition of $\Omega(\mathbf{q},\mathbf{p})$ is independent of the choice of the group element g that moves $(\mathbf{0},\mathbf{0})$ to $(\mathbf{q},\mathbf{p})$, in other words, Ω verifies the covariance property as we can see in the following way: the isotopy group of $(\mathbf{0},\mathbf{0})$ is

$$\Gamma_{(0,0)} \equiv \{g \in G \mid g = (\theta, b, \mathbf{0}, \mathbf{0}, \phi)\}, \tag{5.70}$$

and the restriction of the representation $U_m^{C_1,C_2}$ of $\overline{NH}$ (4.52) to the isotopy group commutes with $\Omega(\mathbf{0},\mathbf{0})$ as it is easy to check. It was proved in Ref. [3] that this commutation is a necessary and sufficient condition to assure the covariance property.

The kernel Ω also verifies the traciality property, which is checked as follows

$$\begin{aligned}
\mathrm{Tr}[\Omega(\mathbf{0},\mathbf{0})\Omega(\mathbf{q},\mathbf{p})] &= \int_{\mathbb{R}^2} \langle \mathbf{y}|\Omega(\mathbf{0},\mathbf{0})\Omega(\mathbf{q},\mathbf{p})|\mathbf{y}\rangle \; d\mathbf{y} \\
&= 2^4 \int_{\mathbb{R}^2} e^{2i\mathbf{q}(\mathbf{p}-m\mathbf{y})} \langle \mathbf{y}|\mathbf{y} - \frac{2}{m}\mathbf{p}\rangle \; d\mathbf{y} \qquad (5.71) \\
&= 2^2 m^2 \delta(\mathbf{p}) \int_{\mathbb{R}^2} e^{-2im\mathbf{q}\mathbf{y}} \; d\mathbf{y} = \delta(\mathbf{q})\; \delta(\mathbf{p}).
\end{aligned}$$

This property is crucial in the theory because it allows to get an inversion formula for the SW correspondence and to evaluate quantum mean values by integration of corresponding functions along the phase space.

It is also easy to find an invariant measure $d\mu(\mathbf{u})$ on the orbit with expression $d\mu(\mathbf{u}) = d\mathbf{q}\, d\mathbf{p}$ that we use to integrate on the orbit (phase space).

Finally, we conclude quantization process calculating the tri–kernel of the SW correspondence, that turns out to be

$$\mathrm{Tr}[\Omega(\mathbf{q},\mathbf{p})\Omega(\mathbf{q}',\mathbf{p}')\Omega(\mathbf{q}'',\mathbf{p}'')] = 2^4\; e^{2i\mathbf{q}(\mathbf{p}'-\mathbf{p}'')}\; e^{2i\mathbf{q}'(\mathbf{p}''-\mathbf{p})}\; e^{2i\mathbf{q}''(\mathbf{p}-\mathbf{p}')}, \tag{5.72}$$

and provide us with the integral kernel which defines the twisted product.

We must notice here that this scheme of quantization can be accomplished for all kind of orbits associated to $NH(2+1)$ groups, although due to the geometry of some orbits it is necessary to introduce additional machinery related with quantization of cylindric orbits that is out of the scope of this article and we postpone it to subsequent papers.

6 Concluding remarks

We briefly discuss here the significance of the second extension associated with parameter f that appeared in section 3. This extension does not exists

for the groups NH in (1+1) and (3+1) dimensions. In fact, a similar situation appears for (2+1) Galilei group in [8]. As we pointed out in the introduction, this difference is easily understood taking into account that Newton–Hooke groups $NH_{\pm}$ are space–velocity contractions of the de Sitter groups $dS_{\pm}$, i.e., NH groups are to the dS ones as Galilei group is to Poincar one [1, 2].

In table 1 we present the commutation relations for the Lie algebras of the four groups above mentioned (in the second part of the table we show the central extension structure). From them we see that contraction $dS_{\pm} \longrightarrow NH_{\pm}$ is accomplished in the limit $c \to \infty$, $R \to \infty$ while the quotient c/R is maintained constant (c can be interpreted as the light speed and R as the curvature radius of the de Sitter universe). In this sense, the

de Sitter	Newton–Hooke	Galilei	Poincar
$[J,K_i] = \epsilon_{ij}K_j$	$[J,K_i] = \epsilon_{ij}K_j$	$[J,K_i] = \epsilon_{ij}K_j$	$[J,K_i] = \epsilon_{ij}K_j$
$[J,P_i] = \epsilon_{ij}P_j$	$[J,P_i] = \epsilon_{ij}P_j$	$[J,P_i] = \epsilon_{ij}P_j$	$[J,P_i] = \epsilon_{ij}P_j$
$[J,H] = 0$	$[J,H] = 0$	$[J,H] = 0$	$[J,H] = 0$
$[K_i,K_j] = -\epsilon_{ij}\frac{1}{c^2}J$	$[\mathbf{K_i},\mathbf{K_j}] = \mathbf{0}$	$[\mathbf{K_i},\mathbf{K_j}] = \mathbf{0}$	$[K_i,K_j] = -\epsilon_{ij}J$
$[K_i,P_j] = \delta_{ij}\frac{1}{c^2}H$	$[\mathbf{K_i},\mathbf{P_j}] = \mathbf{0}$	$[\mathbf{K_i},\mathbf{P_j}] = \mathbf{0}$	$[K_i,P_j] = \delta_{ij}H$
$[K_i,H] = P_i$	$[K_i,H] = P_i$	$[K_i,H] = P_i$	$[K_i,H] = P_i$
$[P_i,P_j] = \mp\epsilon_{ij}\frac{1}{R^2}J$	$[\mathbf{P_i},\mathbf{P_j}] = \mathbf{0}$	$[P_i,P_j] = 0$	$[P_i,P_j] = 0$
$[P_i,H] = \mp(\frac{c}{R})^2K_i$	$[P_i,H] = \mp\frac{1}{\tau^2}K_i$	$[P_i,H] = 0$	$[P_i,H] = 0$
trivial extensions	$[K_i,P_j] = \delta_{ij}M$ $[K_i,K_j] = \epsilon_{ij}F$ $[P_i,P_j] = \epsilon_{ij}\frac{1}{\tau^2}F$ $([H,J] = L)$	$[K_i,P_j] = \delta_{ij}M$ $[K_i,K_j] = \epsilon_{ij}F$ $([H,J] = L)$	trivial extensions

Table 1: Commutation relations in (2+1) dimensions.

interpretation of extension f is parallel to that made for Galilei group in [8], where it is showed that f is associated to a non–relativistic residue of non–commutativity of boosts ($[K_1,K_2] = -J$) for Poincar group, which physically

would corresponds to a non–relativistic *Thomas precession* [32]. It was also noticed that extension f is essentially different from the one corresponding to the mass of the system (m).

However, it seems difficult, from the physical point of view, that such *precession* could be a measurable magnitude. We introduce a new perspective in favor of this argument; our reasoning is related with the quantization process we exposed in previous section, and it departs from the fact that if we use a set of suitable canonical coordinates for the orbits with $(f, m \neq 0)$ and $(f = 0, m \neq 0)$, the parameter f in the first case can be embedded in the definition of *position* $\mathbf{q}$, but "disappear" in the expressions of the kernel and trikernel of the SW correspondence, which become identical to their analogues for the second case.

This result is in concordance with the ideas exposed in [33] about the existence of an isomorphism between all the extended algebras with $m \neq 0$ for arbitrary f, and give a new point of view to the hypothesis that extension linked to f is not physically very relevant in order to study the corresponding elementary systems.

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